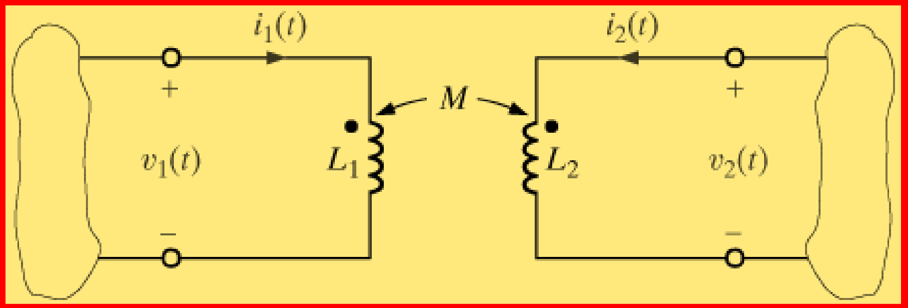
When the **voltages and currents are sinusoidal,** we work in the **phasor domain** where the coupling equations become:



$$
\begin{aligned}
V_1 &= j\omega L_1 I_1 + j\omega M I_2 \\
V_2 &= j\omega M I_1 + j\omega L_2 I_2
\end{aligned}
$$

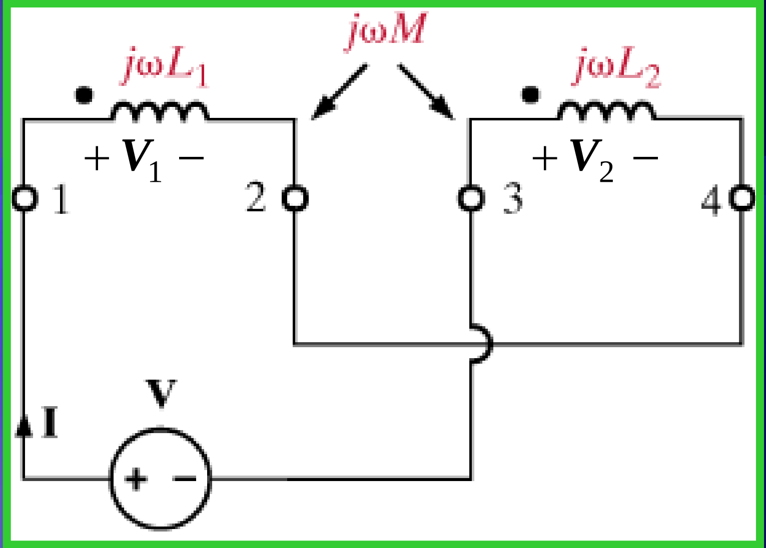
Example 10.3 – Equivalent inductance of two mutually coupled coils (Cases 2 and 3)

CASE 2

$V = V_1 - V_2$

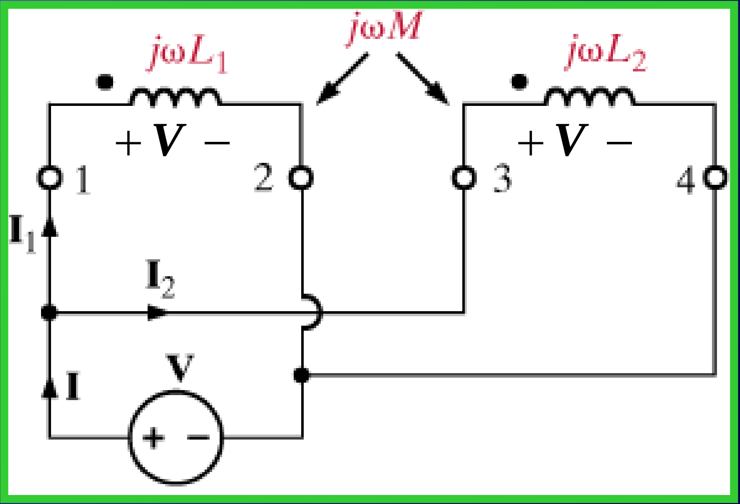
$$
\begin{aligned}
V_1 &= j\omega L_1 - j\omega M I \\
V_2 &= j\omega M I - j\omega L_2 I
\end{aligned}
$$

$$V = j\omega\overbrace{(L_1 - 2M + L_2)}^{L_{eq}}I$$



$L_{eq} \geq 0$ imposes a **physical constraint** on the value of M.

CASE 3



$$I = I_1 + I_2 \Rightarrow I_2 = I - I_1$$

$$\boldsymbol{V = j\omega L_1 I_1 + j\omega M I_2}$$
$$\boldsymbol{V = j\omega M I_1 + j\omega L_2 I_2}$$

$$\boldsymbol{V = j\omega L_1 I_1 + j\omega M (I - I_1)}$$
$$\boldsymbol{V = j\omega M I_1 + j\omega L_2 (I - I_1)}$$

$$\boldsymbol{V = j\omega (L_1 - M) I_1 + j\omega M I} \quad \times / (L_2 - M)$$
$$\boldsymbol{V = -j\omega (L_2 - M) I_1 + j\omega L_2 I} \quad \times / (L_1 - M)$$

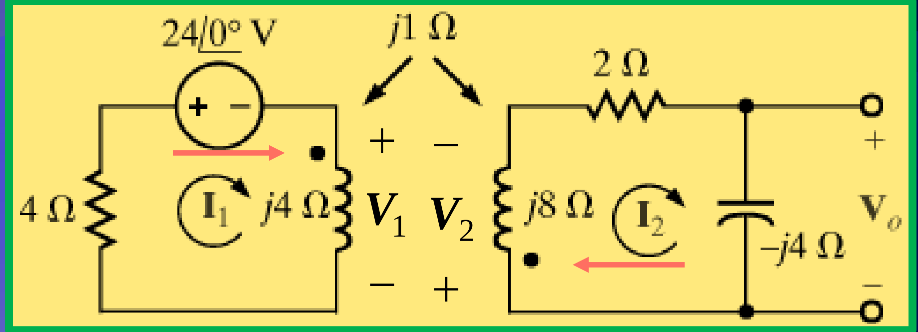
$$\boldsymbol{(L_1 + L_2 - 2M) V = j\omega \big(M (L_2 - M) + L_2 (L_1 - M)\big) I}$$

$$V = j\omega \underbrace{\frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}}_{L_{eq}} I$$

Example 10.4 – Output voltage of a circuit with two coupled coils

Example 10.5 – Mesh equations of a three-mesh circuit with 2 coupled coils

E 10.2 – Currents and output voltage in a circuit with two coupled coils



1. Define variables for coupled inductors

2. Loop equations:

$$\boldsymbol{V_S + V_1 + 4 I_1 = 0}$$

$$\boldsymbol{V_2 + (2 - j4) I_2 = 0}$$

3. Coupled inductors equations:

$$\boldsymbol{V_1 = j4 I_1 + j I_2}$$
$$\boldsymbol{V_2 = j I_1 + j8 I_2}$$

4. Replace and rearrange:

$$\boldsymbol{(4 + j4) I_1 + j I_2 = -V_S}$$
$$\boldsymbol{j I_1 + (2 + j4) I_2 = 0}$$

$$\Rightarrow (1 + 8(1 + j)(1 + 2j)) I_2 = j V_S$$

4. Deduce $I_{2,}$ $I_{1,}$ and $V_{0,}$:

$$I_2 = \frac{jV_S}{-7+24j} \times \frac{-j}{-j} = \frac{24\angle 0^\circ}{24+7j} = \frac{24\angle 0^\circ}{25\angle 16.26^\circ}$$

$$I_2 = 0.96\angle -16.26^\circ (A)$$

$$jI_1 + (2+j4)I_2 = 0 / \times j \Rightarrow I_1 = j(2+j4)I_2$$

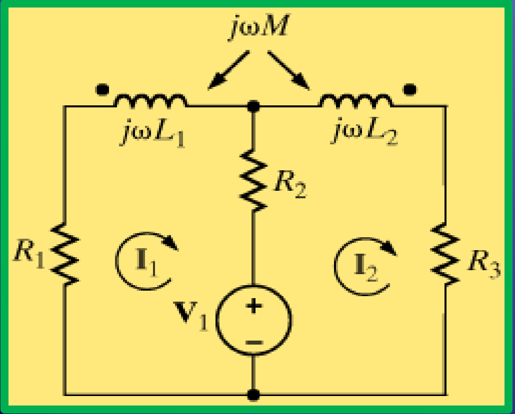
$$I_1 = 1\angle 90^\circ \times 4.47\angle 63.43^\circ \times 0.96\angle -16.26^\circ$$

$$I_1 = 4.29\angle 137.17^\circ (A)$$

$$V_0 = -j4I_2 = 1\angle -90^\circ \times 4 \times 0.96\angle -16.26^\circ$$

$$V_0 = 3.84\angle -106.26^\circ (V)$$

E 10.3 – Two-mesh Circuit



1. Define variables for coupled inductors

2. Loop equations in terms of inductor voltages:

$$V_a + R_2(I_1 - I_2) + V_1 + R_1 I_1 = 0$$

$$-V_b + R_3 I_2 - V_1 + R_2(I_2 - I_1) = 0$$

3. Equations for coupled inductors:

$$V_a = j\omega L_1 I_1 + j\omega M(-I_2)$$

$$V_b = j\omega M I_1 + j\omega L_2(-I_2)$$

4. Replace into loop equations and rearrange:

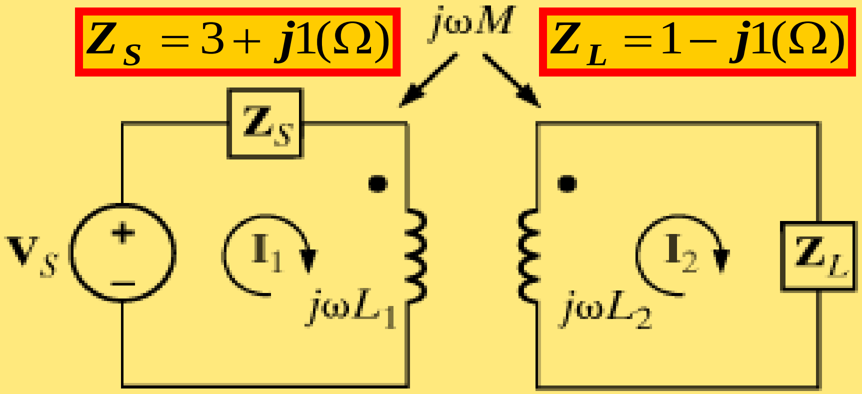
$$(R_1 + R_2 + j\omega L_1)I_1 - (R_2 + j\omega M)I_2 = -V_1$$

$$-(R_2 + j\omega M)I_1 + (R_2 + R_3 + j\omega L_2)I_2 = V_1$$

Example 10.6 – Impedance Seen by Source

$Z_S = 3 + j1(\Omega)$

$Z_L = 1 - j1(\Omega)$



$j\omega L_1 = j2(\Omega)$ $\quad$ $j\omega L_2 = j2(\Omega)$

$j\omega M = j1(\Omega)$

1. Variables for coupled inductors:

2. Loop equations in terms of coupled inductors voltages:

$$Z_S I_1 + V_1 = V_S$$

$$-V_2 + Z_L I_2 = 0$$

3. Equations for coupled inductors:

$$V_1 = j\omega L_1 I_1 + j\omega M(-I_2)$$

$$V_2 = j\omega M I_1 + j\omega L_2(-I_2)$$

4. Replace and do the algebra:

$$(Z_S + j\omega L_1)I_1 - (j\omega M)I_2 = V_S \qquad \times/(Z_L + j\omega L_2)$$

$$-(j\omega M)I_1 + (Z_L + j\omega L_2)I_2 = 0 \qquad \times/\,j\omega M$$

$$\left((Z_S + j\omega L_1)(Z_L + j\omega L_2) - (j\omega M)^2\right)I_1 = (Z_L + j\omega L_2)V_S$$

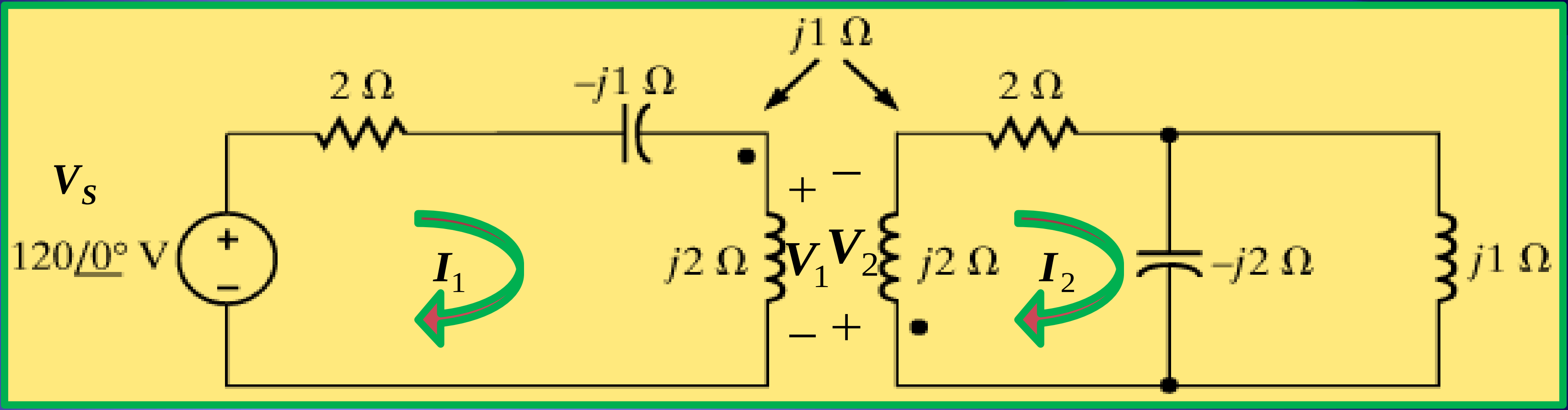
$$Z_i = \frac{V_S}{I_1} = (Z_S + j\omega L_1) - \frac{(j\omega M)^2}{Z_L + j\omega L_2}$$

$$Z_i = 3 + j3 - \frac{(j1)^2}{1 + j1} = 3 + j3 + \frac{1}{1 + j} \qquad \times\frac{1 - j}{1 - j}$$

$$Z_i = 3 + j3 + \frac{1 - j}{2} = 3.5 + j2.5(\Omega)$$

$$Z_i = 4.30\angle 35.54°(\Omega)$$

E 10.4 – Impedance Seen by Source



$\boldsymbol{Z_S} = 2 - \boldsymbol{j}1(\Omega)$

$\boldsymbol{Z_L} = 2 + (-\boldsymbol{j}2 \parallel \boldsymbol{j}1)$

$$\boldsymbol{Z_L} = 2 + \frac{2}{-\boldsymbol{j}} = 2 + 2\boldsymbol{j}(\Omega)$$

1. Variables for coupled inductors

2. Loop equations

$$\boldsymbol{V}_1 + \boldsymbol{Z_S I}_1 = \boldsymbol{V_S}$$
$$\boldsymbol{V}_2 + \boldsymbol{Z_L I}_2 = 0$$

3. Equations for coupled inductors

$$\boldsymbol{V}_1 = \boldsymbol{j}\omega \boldsymbol{L}_1 \boldsymbol{I}_1 + \boldsymbol{j}\omega \boldsymbol{M I}_2$$
$$\boldsymbol{V}_2 = \boldsymbol{j}\omega \boldsymbol{M I}_1 + \boldsymbol{j}\omega \boldsymbol{L}_2 \boldsymbol{I}_2$$

4. Replace and do the algebra

$$\boldsymbol{Z_i} = \frac{\boldsymbol{V_S}}{\boldsymbol{I}_1} = (\boldsymbol{Z_S} + \boldsymbol{j}\omega \boldsymbol{L}_1) - \frac{(\boldsymbol{j}\omega \boldsymbol{M})^2}{\boldsymbol{Z_L} + \boldsymbol{j}\omega \boldsymbol{L}_2}$$

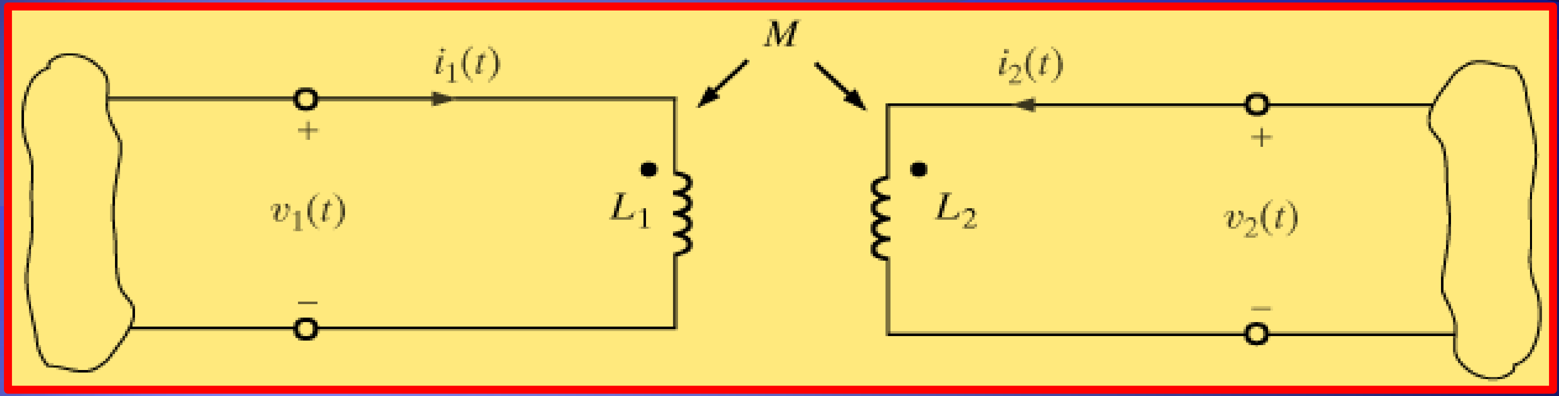
$$\boldsymbol{Z_i} = [(2 - \boldsymbol{j}1) + \boldsymbol{j}2] - \frac{(\boldsymbol{j}1)^2}{(2 + 2\boldsymbol{j}) + 2\boldsymbol{j}}$$

$$= (2 + \boldsymbol{j}) + \frac{1}{2(1 + 2\boldsymbol{j})} \quad \times \frac{(1 - 2\boldsymbol{j})}{(1 - 2\boldsymbol{j})}$$

$$\boldsymbol{Z_i} = 2 + \boldsymbol{j} + \frac{1 - 2\boldsymbol{j}}{2(1 + 2^2)} = 2.1 + 0.8\boldsymbol{j}(\Omega)$$

$$\boldsymbol{Z_i} = 2.25\angle 20.85°(\Omega)$$

ENERGY ANALYSIS



In the following, a small **experiment** will be conducted on the above circuit, where the two coils, L_1 and L_2, are mutually coupled. The experiment will be conducted in **two phases**:

Phase 1:

In this phase, we leave the **right side terminal open**, i.e. $i_2(t) = 0$, and we increase the current **$i_1(t)$ from 0 to some value I_1**. In this case, whereas the power associated with the right side will be zero, the **instantaneous power at the left-side terminals** could be calculated as:

$$\boxed{p(t) = v_1(t)i_1(t) = \left[L_1\frac{di_1}{dt}\right]i_1(t)}$$

When **$i_1(t)$ reaches I_1** at a certain time, t_1, **the energy stored within the coupled circuit** could be found as:

$$\boxed{\int_0^{t_1} v_1(t)i_1(t)dt = \int_0^{I_1} L_1 i_1(t)di_1(t) = \frac{1}{2}L_1 I_1^2}$$

Phase 2:

In this phase, while **holding $i_1(t)$ constant at I_1**, we increase the current **$i_2(t)$ from 0 to a certain value I_2**. Consequently, we could calculate the **energy associated with the right-side terminals**:

$$\int_{t_1}^{t_2} v_2(t) i_2(t) dt = \int_0^{I_2} L_2 i_2(t) di_2(t) = \frac{1}{2} L_2 I_2^2$$

At the same time, it is possible to calculate **the voltage and the energy at the left-side terminals**:

$$v_1(t) = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = M \frac{di_2}{dt}$$

since $i_1(t)$ remains constant at I_1, and:

$$\int_{t_1}^{t_2} v_1(t) i_1(t) dt = \int_{t_1}^{t_2} M \frac{di_2}{dt} I_1 dt$$
$$= M I_1 \int_0^{I_2} di_2 = M I_1 I_2$$

The **total energy stored** in the circuit from time zero until time t_2 passing by time t_1 is:

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$$

If the dot on one of the coils is reversed, however, we could calculate the total energy to be:

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 - M I_1 I_2$$

Also, **for an arbitrary time, t,** the expression of the total energy could be shown to be:

$$w(t) = \frac{1}{2} L_1 [i_1(t)]^2 + \frac{1}{2} L_2 [i_2(t)]^2 \pm M i_1(t) i_2(t)$$

Since the two coils represent a passive network, this energy must by nonnegative:

To the expression of the instantaneous energy, we can **add and subtract** the following term:

$$\boxed{\frac{1}{2}\frac{M^2}{L_2}i_1^2}$$

After rearranging, we find the following expression:

$$\boxed{w(t)=\frac{1}{2}\left(L_1-\frac{M^2}{L_2}\right)i_1^2+\frac{1}{2}L_2\left(i_2+\frac{M}{L_2}i_1\right)^2}$$

Since the second term in the above equation is **a square**, we deduce that the condition for $w(t)$ to be nonnegative is:

$$\boxed{M\le\sqrt{L_1L_2}}$$

which specifies an **upper limit on the value of *M*.**

If we define the **coefficient of coupling** as:

$$\boxed{k=\frac{M}{\sqrt{L_1L_2}}}$$

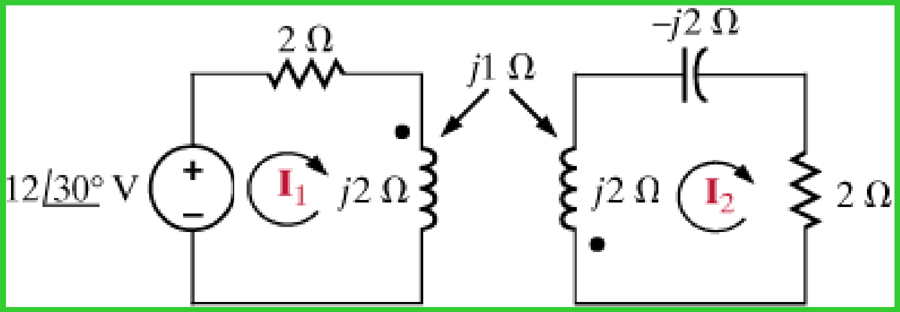
The **condition on the coefficient of coupling** could be stated as:

$$\boxed{0\le k\le 1}$$

Hence, this coefficient indicates **how much flux in one coil affects the other coil** , and vice versa. For $k = 0$, there is **no coupling** between the two coils, and for $k = 1$, the two coils are said to be **fully coupled**. In this respect, **if $k > 0.5$** the two coils are said to be **tightly coupled**, whereas **if $k < 0.5$** the two coils are said to be **loosely coupled**.

Example 10.7 – Calculation of Stored Energy

E 10.5 – Calculation of Stored Energy in Mutually Coupled Coils



$$f = 60Hz \Rightarrow \omega = 377(s^{-1})$$

$$\omega \boldsymbol{L}_1 = 2 \Rightarrow \boldsymbol{L}_1 = 0.00528(\boldsymbol{H}) = \boldsymbol{L}_2$$

$$\boldsymbol{M} = 0.00264(\boldsymbol{H})$$

$$2\boldsymbol{I}_1 + (\boldsymbol{j}2\boldsymbol{I}_1 + \boldsymbol{j}1\boldsymbol{I}_2) = 12\angle 30°$$

$$(\boldsymbol{j}1\boldsymbol{I}_1 + \boldsymbol{j}2\boldsymbol{I}_2) + (2 - \boldsymbol{j}2)\boldsymbol{I}_2 = 0$$

$$(2+\boldsymbol{j}2)\boldsymbol{I}_1 + \boldsymbol{j}\boldsymbol{I}_2 = 12\angle 30°$$

$$\boldsymbol{j}\boldsymbol{I}_1 + 2\boldsymbol{I}_2 = 0$$

$$\Rightarrow \boldsymbol{I}_2 = -0.5\boldsymbol{j}\boldsymbol{I}_1$$

$$(2+\boldsymbol{j}2+0.5)\boldsymbol{I}_1 = 12\angle 30°$$

$$\boldsymbol{I}_1 = \frac{12\angle 30°}{2.5+\boldsymbol{j}2} = \frac{12\angle 30°}{3.20\angle 38.66°}$$

$$= 3.75\angle -8.66°(\boldsymbol{A})$$

$$\boldsymbol{I}_2 = -0.5\boldsymbol{j}\boldsymbol{I}_1 = 0.5\angle -90° \times 3.75\angle -8.66°$$
$$= 1.875\angle -98.66°$$

Go back to time domain:

$$i_1(t) = 3.75\cos(377t - 8.66°)(A)$$
$$i_2(t) = 1.875\cos(377t - 98.66°)(A)$$

$$377(rad/\sec) \times 0.010(\sec) = 3.77(rad) = 216°$$

$$i_1(0.010) = -3.3(A) \text{ and } i_2(0.010) = -0.86(A)$$

$$w(t) = \frac{1}{2}L_1 i_1^2(t) + M i_1(t) i_2(t) + \frac{1}{2}L_2 i_2^2(t)$$

$$w(0.010) = 0.5 * 0.00528 * (-3.3)^2$$
$$+ 0.00264 * (-3.3)(-0.86)$$
$$+ 0.5 * 0.00528 * (0.86)^2 (J)$$

$$w(0.010) = 0.00264 * (3.3^2 + (3.3)(0.86) + 0.86^2)$$

$$w(0.010) = 0.038J = 38mJ$$