CHAPTER 10 – ANALYSIS OF MAGNETICALLY COUPLED CIRCUITS

LEARNING GOALS

1. To learn how to apply Ampere's law and Faraday's law on magnetically coupled coils

2. To differentiate between self inductance and mutual inductance

3. To learn the relation between mutual inductance and coupling coefficient

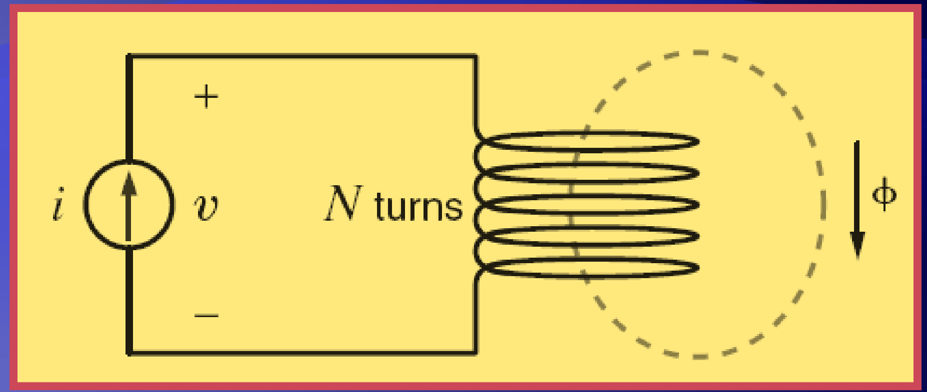
4. To calculate voltages and currents in circuits containing mutual inductance

5. To calculate voltages and currents in circuits containing ideal transformers

INTRODUCTION

Most **kitchen cooktops** used today use **gas or electricity** to heat up **an element**, which then transfers the heat to a pot or pan placed on it. This type of cooktops has an **efficiency of about 40 or 50%** due to the **indirect heat transfer**. Another type of cooktops is the **induction cooktop** which uses an electromagnet excited by a high frequency ac signal . The magnetic field created by the electromagnet induces a **current into the pot or pan and heats it directly**. This type of cooktops has an **efficiency reaching 90%.** In addition, because the pot or pan is heated directly, the **surface of the cooktop remains cool!**

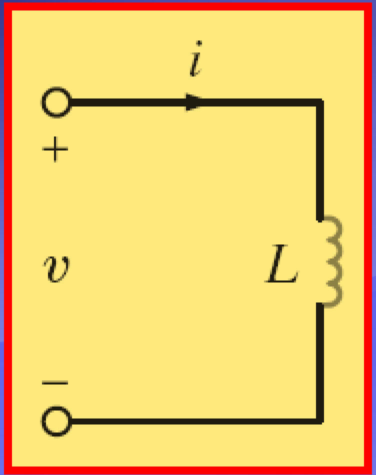
MUTUAL INDUCTANCE



Given a time-varying current, ***i*(*t*)**, flowing in an **ideal coil** that has *N* turns, **Ampere's law** predicts that a **magnetic flux** will be created around the coil, and that the **flux linkage** is proportional to the current that has created it:

$$\boxed{\lambda(t) = N\varphi(t) = Li(t)}$$

Where L is the (self) inductance of the coil, and $\varphi(t)$ is the magnetic flux.

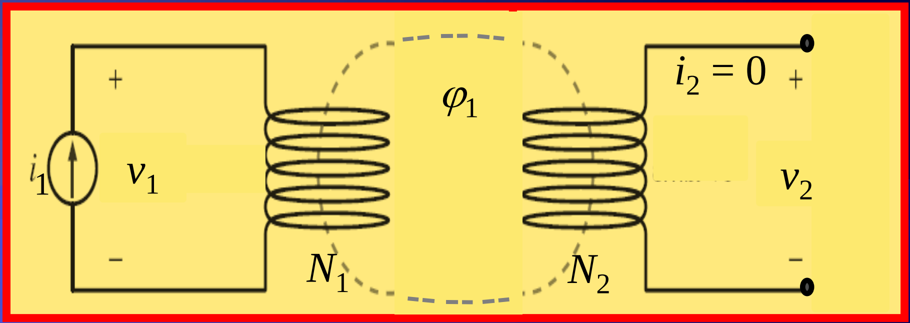


On the other hand, **Faraday's law** predicts that the **flux linkage induces a voltage** across the coil equal to:

$$v(t)=\frac{d\lambda(t)}{dt}=L\frac{di(t)}{dt}$$

where $i(t)$ and $v(t)$ satisfy the **passive sign convention**.

If we now bring a **second ideal, open-ended coil with N_2 turns** close enough to the first coil such that **the flux pro-**



duced by the first coil also links the second coil, we could express the flux linkage of the second coil by:

$$\lambda_2(t)=N_2\varphi_1(t)$$

Where the flux $\varphi_1(t)$ is produced by the current $i_1(t)$ flowing in the first coil.

$$\lambda_1(t)=N_1\varphi_1(t)=L_1 i_1(t)$$

Consequently, **although no current is flowing in the second coil**, a voltage across it will be induced:

$$v_2(t)=\frac{d\lambda_2(t)}{dt}=N_2\frac{d\varphi_1(t)}{dt}$$

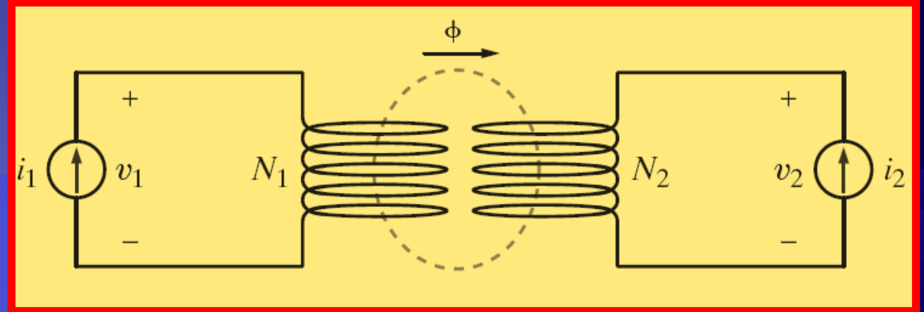
Using the expression of $\varphi_1(t)$ produced by $i_1(t)$:

$$\boxed{\varphi_1(t) = \frac{L_1}{N_1} i_1(t)}$$

we deduce the expression of **the voltage $v_2(t)$ induced by current $i_1(t)$ across the second coil**:

$$\boxed{v_2(t) = \frac{N_2}{N_1} L_1 \frac{di_1(t)}{dt} \equiv L_{21} \frac{di_1(t)}{dt}}$$

As we can see, **the induced voltage across the second coil is proportional to rate of change of the current flowing in the first coil**. The constant of proportionality is called the **mutual inductance** and is expressed in **Henrys**. We say that the two coils are **magnetically coupled.**



At this stage, **if we connect a current source to the terminals of the second coil**, it will not only create a voltage across the second coil, but **it will also induce an additional voltage across the first coil**, due to the **magnetic coupling** between the two coils.

In the case where both currents, **$i_1(t)$ and $i_2(t)$, contribute to the magnetic flux, $\varphi(t)$** (which depends on the respective orientations of the coil windings), the flux linkages for each coil could be expressed as:

$$\lambda_1 = L_1 i_1 + L_{12} i_2$$
$$\lambda_2 = L_2 i_2 + L_{21} i_1$$

And the resulting expressions of the voltages across each coil become:

$$v_1 = \frac{d\lambda_1}{dt} = L_1 \frac{di_1}{dt} + L_{12} \frac{di_2}{dt}$$
$$v_2 = \frac{d\lambda_2}{dt} = L_2 \frac{di_2}{dt} + L_{21} \frac{di_1}{dt}$$

For **linear circuits**, we generally consider that:

$$L_{12} = L_{21} = M$$

Where M is the **mutual inductance**.

We can see that the voltage across each coil is now composed of two terms:

- A **self-induced term** of the form $L_i \dfrac{di_i}{dt}$
- A **mutually-induced term** of the form:

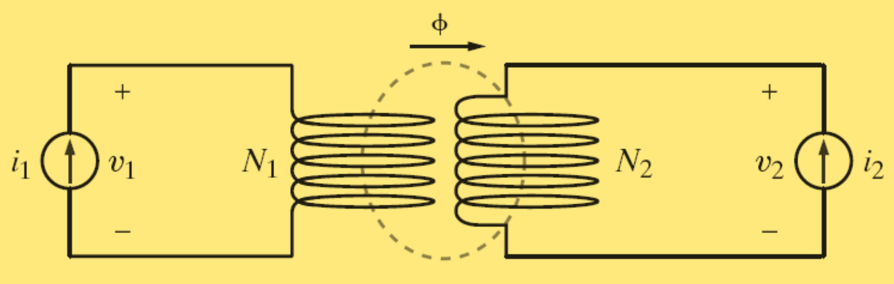
$$M \frac{di_j}{dt}$$

where i and j are 1 and 2.

We note that **if the current $i_2(t)$ is reversed**, its induced flux will oppose the magnetic flux induced by $i_1(t)$. In this case, the expressions of the voltages across each coil become:

$$v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}$$
$$v_2 = L_2 \frac{di_2}{dt} - M \frac{di_1}{dt}$$

THE DOT CONVENTION



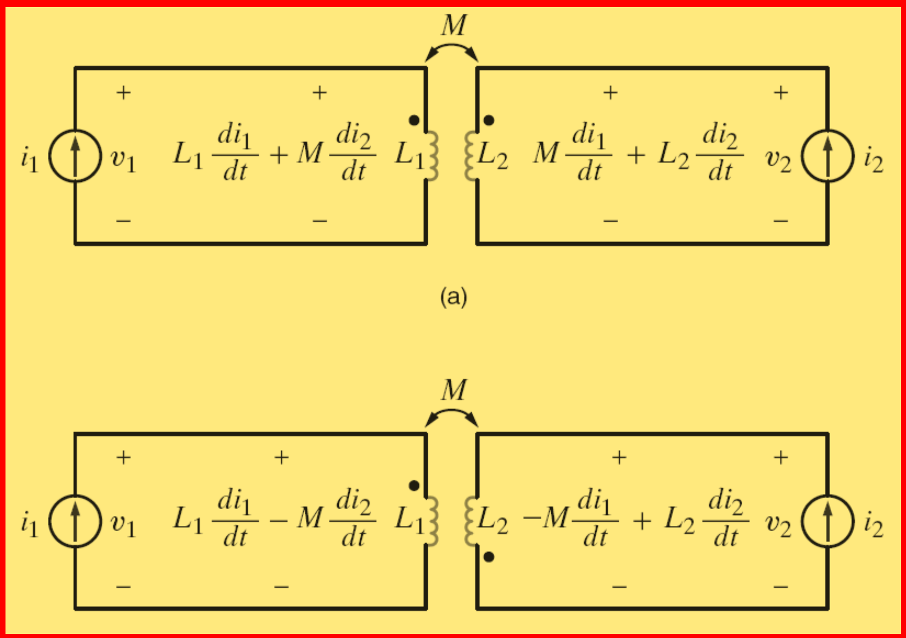
We note that a **negative mutually-induced voltage** will also result if the **orientation of the winding** of one of the coils is **reversed**, as shown in the Figure.

Since the representation of the winding orientation of coils in circuit diagrams may be complicated, we adopt the **dot convention** to take into consideration the **winding orientation** of the coils.

To each coil we associate a dot placed on one of its terminals such that **if the current flows into the dot** of this coil it induces a **mutual voltage** into the second coil that is **positive at the dot of the second coil.** Conversely, **if the current in the first coil flows out of its dot**, it induces into the second coil a **voltage that is negative at the dot of the second coil**.

It is to be noted that:

- the **self-induced voltage follows the passive sign convention**
- while the **mutually-induced voltage follows the dot convention**.



In the first circuit diagram above, **both currents flow into the dots** of their respective coils. Therefore, the circuit diagram is a representation of the case where the current **$i_1(t)$ contributes to the flux produced by $i_2(t)$,** and vice-versa, i.e. the mutually-induced voltage produced by $i_1(t)$ across coil L_2 is positive at the dot of coil L_2 and vice-versa.

However, in the second circuit diagram the current in coil L_1 flows into its dot while the **current in coil L_2 flows out of its dot**. The circuit diagram is therefore a representation of the case where the current **$i_1(t)$ produces a flux that is opposed to the flux produced by $i_2(t)$,** and vice versa, i.e. the mutually-induced voltage produced by $i_1(t)$ across coil L_2 is negative at the dot of coil L_2 and vice-versa.

PROBLEM SOLVING STRATEGY

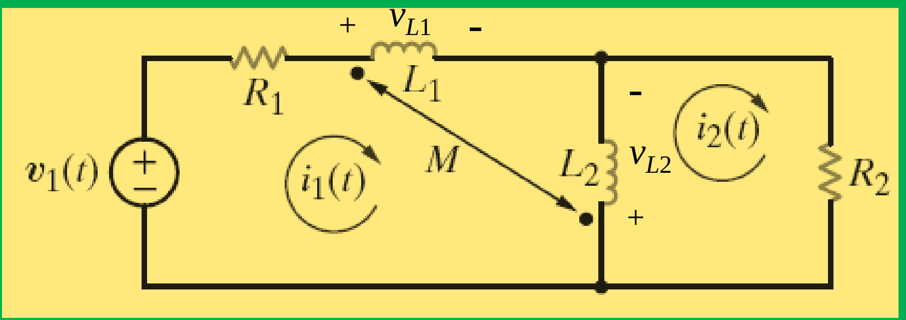
Step 1: Assign **mesh currents** and **coil voltages** and write **coupling equations** using dot convention

Step2: Write **mesh equations** using mesh currents and coil voltages

Step3: **Solve** for currents and voltages.

Example 10.1 – Determination of coupling equations

Example 10.2 – Mesh equations for magnetically coupled and electrically connected coils



Mesh equations:

$$-v_1(t)+R_1i_1(t)+v_{L1}(t)-v_{L2}(t)=0$$
$$v_{L2}(t)+R_2i_2(t)=0$$

Coupling equations:

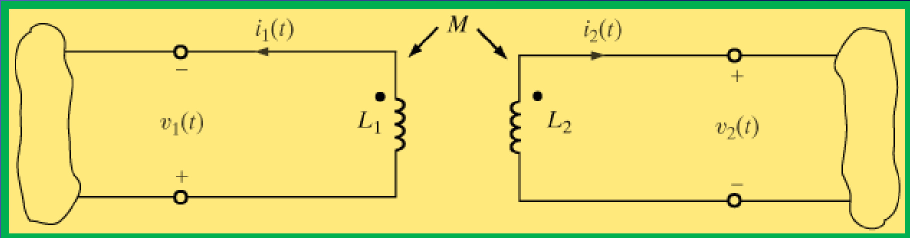
$$v_{L1}(t)=L_1\frac{di_1(t)}{dt}+M\frac{d(i_2(t)-i_1(t))}{dt}$$
$$v_{L2}(t)=L_2\frac{d(i_2(t)-i_1(t))}{dt}+M\frac{di_1(t)}{dt}$$

Resulting equations:

$$v_1(t)=R_1i_1(t)+L_1\frac{di_1(t)}{dt}+M\frac{d(i_2(t)-i_1(t))}{dt}-L_2\frac{d(i_2(t)-i_1(t))}{dt}-M\frac{di_1(t)}{dt}$$

$$L_2\frac{d(i_2(t)-i_1(t))}{dt}+M\frac{di_1(t)}{dt}+R_2i_2(t)=0$$

Exercise E10.1 – Coupling Equations for magnetically coupled coils



$$v_1(t)=L_1\frac{di_1(t)}{dt}+M\frac{di_2(t)}{dt}$$
$$v_2(t)=-L_2\frac{di_2(t)}{dt}-M\frac{di_1(t)}{dt}$$